

# Non existence of vortices in the small density region of a condensate

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## Abstract

In this paper, we answer a question raised by Len Pitaevskii and prove that the ground state of the Gross Pitaevskii energy describing a Bose Einstein condensate at low rotation does not have vortices in the low density region. Therefore, the first ground state with vortices has its vortices in the bulk. This is obtained by proving that for small rotational velocities, the ground state is multiple of the ground state with zero rotation. We rely on sharp bounds of the decay of the wave function combined with weighted jacobian estimates.

## 1 Introduction

Among the many experiments on Bose Einstein condensates, one consists in rotating the trap holding the atoms in order to observe a superfluid behaviour: the appearance of quantized vortices [1, 23, 19, 20, 2]. This takes place for sufficiently large rotational velocities. On the contrary, at low rotation, no vortex is detected in the bulk of the condensate. The system can be described by a complex valued wave function minimizing a Gross Pitaevskii type energy. A vortex corresponds to zeroes of the wave function with phase around it. The density of the condensate is significant in a region which is either a disk or an annulus, and gets exponentially small outside this domain. Vortices are experimentally visible in the bulk of the condensate. A question raised by Len Pitaevskii is whether for small rotational velocity, when there are no vortices in the bulk, vortices could exist in the low density region. For very large rotational velocities, when bulk vortices are arranged on a triangular lattice, it has been shown [4] that the vortex distribution extends to infinity. Therefore, in this case, there are many vortices in the low density region. It is then very natural to wonder whether vortices first appear in the bulk or at infinity. It is experimentally and numerically difficult to observe a vortex, which is a zero, in a low density region. Mathematically this could not be achieved through energy estimates or expansion since the contribution of a vortex in a

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low density region is very small. In this paper, we introduce new ideas to answer Pitaevskii's question and prove that at low velocity, there are indeed no vortices in the condensate, even in the low density region. Therefore, the first ground state with vortices has its vortices in the bulk.

Since a condensate is a trapped object, the geometry of the trap plays a role. An important special case is a radial harmonic trapping potential  $V(r) = r^2$ . The space can then be split into two regions, a region of the form  $\mathcal{D} = \{\lambda_0 > V(r)\}$  (for a suitable constant  $\lambda_0$ ), where the wave function is significant and the condensate is mainly located, and a region  $\mathbf{R}^2 \setminus \mathcal{D}$  where the modulus of the wave function is exponentially small [2]. In this latter region, it is very difficult to determine mathematically the contribution of a vortex to the energy. Ignat and Millot [11, 12] have determined the critical rotational velocity  $\Omega_c$  for the nucleation of the first vortex inside  $\mathcal{D}$ . This theorem does not describe the behaviour in  $\mathbf{R}^2 \setminus \mathcal{D}$ . A natural question is whether for  $\Omega < \Omega_c$ , the minimizer of the energy has zeroes in this region, whether there is a smaller critical velocity than  $\Omega_c$  where the minimizer is unique and vortex free. At very high velocity, it has been proved in [4] that vortices exist up to infinity so it seems reasonable that at smaller velocity, vortices may exist in the exponentially small region, far away from the bulk and could arrange themselves on disks or arrays close to infinity. In fact, we prove that this is not the case before  $\Omega_c$ , namely that the minimizer is unique and does not vanish. It means that for a large range of rotational velocities  $\Omega$ , the minimizer is given by the same function.

We consider here a two-dimensional setting and define the energy for the complex-valued wave function  $u$ , such that  $\int_{\mathbf{R}^2} |u|^2 = 1$ , as

$$E_\varepsilon(u) = \int_{\mathbf{R}^2} \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} |u|^4 + \frac{1}{2\varepsilon^2} V(x) |u|^2 - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx, \quad (1.1)$$

where  $\Omega$  is the angular velocity,  $x = (x_1, x_2)$ ,  $x^\perp = (-x_2, x_1)$ ,  $\varepsilon > 0$  is a small parameter,  $V(x)$  is the trapping potential and  $(iu, \nabla u) = iu \nabla u^* - iu^* \nabla u$ . The class of potentials includes the model case  $V = x_1^2 + x_2^2$ . Then, the critical angular velocity for nucleation of vortices is of order  $|\log \varepsilon|$  (see [11]). An upper bound on the rotational velocity is given by  $\Omega < 1/\varepsilon$  when the confinement breaks down. The condensate is mostly concentrated in the region

$$\mathcal{D} := \{x \in \mathbf{R}^2 : V < \lambda_0\} \quad (1.2)$$

where  $\lambda_0$  is chosen so that

$$\int_{\mathbf{R}^2} (\lambda_0 - V(x))^+ dx = 1. \quad (1.3)$$

We refer to [2] for more details on how this is derived from the physical experiments.

In recent experiments in which a laser beam is superimposed upon the magnetic trap holding the atoms, the trapping potential  $V(x)$  is of a different type [21, 23, 24, 5]:

$$V(r) = r^2 + V_0 e^{-r^2/w_0}. \quad (1.4)$$

When the gaussian is expanded around the origin, this leads to a harmonic plus quartic potential [23]

$$V(r) = (1 - b)r^2 + \frac{k}{4}r^4. \quad (1.5)$$

If  $b$  is small ( $b < 1 + (3k^2/4)^{1/3}$ ), the domain  $\mathcal{D}$  given by (1.2) is a disc, while if  $b > 1 + (3k^2/4)^{1/3}$ , it is an annulus. According to the values of  $V_0$  and  $w_0$  in the case of (1.4), the domain  $\mathcal{D}$  can also be a disk or an annulus.

In this paper, we consider potentials  $V$  including  $r^2$  and of the type (1.4) or (1.5) when the bulk  $\mathcal{D}$  is a disk. In the case where  $\mathcal{D}$  is a disk, the potential  $V$  is not necessarily required to be increasing.

## 1.1 Assumptions

Throughout this paper, we make the following assumptions about the potential  $V$ . First,

$$V \text{ is nonnegative and radial, } V \in C^1, \quad (1.6)$$

and

$$\text{there exists } c_0 > 0, p \geq 2 \text{ such that } \frac{1}{c_0} r^p \leq V(r) \leq c_0 r^p \quad \text{if } r \geq c_0. \quad (1.7)$$

This assumption is easily seen to imply that  $E_\varepsilon$  is bounded below for  $|\Omega| \ll \frac{1}{\varepsilon}$  and that the angular momentum term  $x^\perp \cdot (iu, \nabla u)$  is integrable as long as  $u$  has finite energy. We will also use (1.7) to obtain decay estimates that justify for example the integration by parts leading to a decoupling of the energy. We fix  $\lambda_0 \in \mathbf{R}$  such that (1.2)-(1.3) hold. Such a  $\lambda_0$  exists due to the growth of  $V$ . We further assume that the bulk  $\mathcal{D}$  is a disk and not an annulus, that is  $V$  is such that

$$\mathcal{D} = B_R(0) \text{ for some } R > 0 \quad (1.8)$$

and that there exist  $\delta_0 > 0$  and a  $C^1$  function  $R : (-2\delta_0, 2\delta_0) \rightarrow \mathbf{R}$  also denoted  $R_\delta = R(\delta)$ , such that

$$R_0 = R, \quad \{x : V(x) < \lambda_0 + \delta\} = B_{R_\delta}(0) \quad (1.9)$$

$$\text{and } 0 < \frac{1}{C} \leq dR/d\lambda \leq C \text{ for some constant } C$$

where  $B_r(y)$  denotes the open ball of radius  $r$  about  $y$ . This implies that  $\lambda_0 - V$  is bounded away from 0 in the interior of  $\mathcal{D}$ ; in physical terms, this assumption rules out the case of annular bulks and “giant vortices” at low angular velocities. We remark that the assumption above implies that if  $|x| \in (R_{-\delta}, R_\delta)$  and  $0 \leq \delta \leq \delta_0$  then  $\text{dist}(x, \partial\mathcal{D}) = \mathcal{O}(\delta)$ .

We point out that assumptions (1.7) and (1.9) imply that

$$\text{there exists } c_1 > 0 \text{ such that } V(r) - \lambda_0 \geq c_1(r^2 - R^2) \text{ for all } r \geq R. \quad (1.10)$$

Our assumptions include indeed potentials like  $r^2$  or (1.5) for a disk case, and do not require  $V$  to be increasing.

## 1.2 Main result

Our main result is

**Theorem 1.1.** *Assume that  $u_\varepsilon$  minimizes  $E_\varepsilon(\cdot)$  with rotation  $\Omega$ , and let  $\eta_\varepsilon$  denote the minimizer of  $E_\varepsilon(\cdot)$  for  $\Omega = 0$ . There exists  $\varepsilon_0, \omega_0, \omega_1 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$  and  $|\Omega| \leq \omega_0 |\log \varepsilon| - \omega_1 \log |\log \varepsilon|$  then  $u_\varepsilon = e^{i\alpha} \eta_\varepsilon$  in  $\mathbf{R}^2$  for some constant  $\alpha$ .*

In the pure quadratic case  $V = r^2$ , Ignat and Millot [11, 12] have shown that the *bulk* of the condensate (that is any domain contained in  $\mathcal{D}$ ) is vortex-free for  $|\Omega| \leq \omega_0 |\log \varepsilon| - \omega_1 \log |\log \varepsilon|$ , for some  $\omega_1 > 0$  and the same constant  $\omega_0$  that we find in Theorem 1.1. They have no information on what happens in  $\mathbf{R}^2 \setminus \mathcal{D}$ . Our theorem proves that vortices do not lie in  $\mathbf{R}^2 \setminus \mathcal{D}$ . They have also shown that there exists  $\delta > 0$  such that the ground state has at least one vortex in the bulk if  $|\Omega| \geq \omega_0 |\log \varepsilon| + \delta \log |\log \varepsilon|$ . In this sense, our estimate  $|\Omega| \leq \omega_0 |\log \varepsilon| - \omega_1 \log |\log \varepsilon|$  captures the sharp leading-order term, and the correct scaling of the next-order term, of the critical velocity for vortex formation. We point out that our arguments also deal with more general potentials. The arguments used in [11] to prove the existence of interior vortices for rotations greater than  $\omega_0 |\log \varepsilon| + \delta \log |\log \varepsilon|$  should extend with few changes to the more general potentials considered here, using results about auxiliary functions that we establish in Section 3 in place of parallel results from [11]. Thus the constant  $\omega_0$  should also be sharp for these more general potentials.

We split the proof into two independent results. The first main result of this paper asserts roughly speaking that *symmetry breaking occurs first in the interior of  $\mathcal{D}$* : if  $\Omega$  is small enough that there are no vortices in  $\mathcal{D}$ , then there are no vortices anywhere.

**Theorem 1.2.** *Assume that  $u_\varepsilon$  minimizes  $E_\varepsilon(\cdot)$  with rotation  $\Omega$ , and let  $\eta_\varepsilon$  denote the minimizer of  $E_\varepsilon(\cdot)$  for  $\Omega = 0$ . Assume also that  $|\Omega| \leq C |\log \varepsilon|$  for some  $C$ .*

*There exists  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$  and  $\Omega$  is subcritical in the sense that*

$$|u_\varepsilon| \geq \frac{1}{2} \eta_\varepsilon \text{ in } \mathcal{D}_1 := \{x \in \mathcal{D} : \text{dist}(x, \partial \mathcal{D}) \geq |\log \varepsilon|^{-3/2}\} \quad (1.11)$$

*then  $u_\varepsilon = e^{i\alpha} \eta_\varepsilon$  in  $\mathbf{R}^2$  for some constant  $\alpha$ .*

Our second main theorem gives an estimate for the critical value of  $\Omega$ . The statement of the theorem refers to an auxiliary function  $f_0$ : let

$$a(x) = \lambda_0 - V(x), \quad \eta_0 := \sqrt{a^+}, \quad \xi_0(r) = \int_r^\infty s \eta_0^2(s) ds, \quad f_0(r) = \begin{cases} 0 & \text{if } r \geq R \\ \xi_0(r)/\eta_0^2(r) & \text{if } r \leq R; \end{cases} \quad (1.12)$$

**Theorem 1.3.** *Let  $\omega_0 = \frac{1}{2\|f_0\|_\infty}$ . There exists  $\omega_1 > 0$  and  $\varepsilon_1 > 0$  such that if  $|\Omega| \leq \omega_0 |\log \varepsilon| - \omega_1 \log |\log \varepsilon|$  and  $0 < \varepsilon < \varepsilon_1$ , then  $\Omega$  is subcritical in the sense of (1.11), and the conclusion of Theorem 1.2 thus holds.*

In our proof of Theorem 1.3, as in estimates of the critical rotation in works such as [11] and [3], a main point is to obtain sharp energy lower bounds. In all earlier works that we know of, this is done using the vortex ball construction originally introduced by [13] and [22]. In our proof of Theorem 1.3, we avoid any explicit<sup>1</sup> mention of vortex balls by

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<sup>1</sup>However, the proof of Lemma 4.1, see Lemma 8 in [14], ultimately relies on a vortex ball construction appearing in [15].

instead appealing to a result from [14], stated here as Lemma 4.1. This makes our argument considerably shorter than those in [3, 11] and other references.

We point out that the results of [11, 12] do *not* directly imply that Theorem 1.3 holds in the case  $V = r^2$ , although it is possible that this conclusion can be extracted with relatively little effort from arguments in these references.

### 1.3 Main ideas of the proof

The energy minimizers with  $\Omega = 0$  provide real solutions to the Euler-Lagrange equations: when  $\Omega = 0$ ,  $E_\varepsilon(\eta) = G_\varepsilon(\eta)$ , where

$$G_\varepsilon(\eta) = \int_{\mathbf{R}^2} \left\{ \frac{1}{2} |\nabla \eta|^2 + \frac{1}{4\varepsilon^2} |\eta|^4 + \frac{1}{2\varepsilon^2} V(x) |\eta|^2 \right\} dx. \quad (1.13)$$

Our main goal consists in proving that up to the critical velocity of nucleation of bulk vortices, the minimizer of  $E_\varepsilon$  with velocity  $\Omega$  is in fact equal to  $\eta_\varepsilon$ .

The minimizer  $\eta_\varepsilon$  of  $G_\varepsilon$  under the  $L^2$  constraint of norm 1, is (up to a complex multiplier of modulus one) the unique positive solution of

$$-\Delta \eta_\varepsilon + \frac{1}{\varepsilon^2} \eta_\varepsilon (V(x) + \eta_\varepsilon^2) = \frac{1}{\varepsilon^2} \lambda_\varepsilon \eta_\varepsilon \quad (1.14)$$

where  $\frac{1}{\varepsilon^2} \lambda_\varepsilon$  is the Lagrange multiplier, which is also necessarily unique. Moreover,  $\lambda_\varepsilon \rightarrow \lambda_0$ , and  $\eta_\varepsilon^2$  converges to  $a^+$  in  $L^2(\mathcal{D})$  and uniformly on any compact set of  $\mathcal{D}$ . We will need some estimates on the decay of  $\eta_\varepsilon$  at infinity that we prove in section 2.

By a remarkable identity (see Lassoued & Mironescu [17]), for any  $u$ , the energy  $E_\varepsilon$  for any  $\Omega$  splits into two parts, the energy  $G_\varepsilon(\eta_\varepsilon)$  of the density profile and a reduced energy of the complex phase  $v = u/\eta_\varepsilon$ :

$$E_\varepsilon(u) = G_\varepsilon(\eta_\varepsilon) + F_\varepsilon(v) \quad (1.15)$$

$$\text{where } F_\varepsilon(v) = \int_{\mathbf{R}^2} \left\{ \frac{\eta_\varepsilon^2}{2} |\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 - \eta_\varepsilon^2 \Omega x^\perp \cdot (iv, \nabla v) \right\} dx. \quad (1.16)$$

In particular the potential  $V(x)$  only appears in  $G_\varepsilon$ . We will recall the proof of (1.15), as well as that of (1.18) below, in Section 3. This kind of splitting of the energy is by now standard in the rigorous analyses of functionals such as  $E_\varepsilon$ .

Next, define

$$\xi_\varepsilon(r) = \int_r^\infty s \eta_\varepsilon^2(s) ds, \quad (1.17)$$

so that  $\nabla^\perp \xi_\varepsilon = x^\perp \eta_\varepsilon^2$ . An integration by parts yields

$$F_\varepsilon(v) = \int_{\mathbf{R}^2} \left\{ \frac{\eta_\varepsilon^2}{2} \left( |\nabla v|^2 - \frac{4\Omega \xi_\varepsilon}{\eta_\varepsilon^2} Jv \right) + \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 \right\} dx \quad (1.18)$$

where  $Jv = \frac{1}{2} \nabla \times (iv, \nabla v) = (iv_{x_1}, v_{x_2})$  is the Jacobian.

We recall that the function  $f_\varepsilon := \xi_\varepsilon/\eta_\varepsilon^2$  appearing in  $F_\varepsilon$  is important since it is well known that vortices in the interior of  $\mathcal{D}$  first appear near where this function attains a local

maximum [2, 3, 11, 12]; its importance is also clear from (1.18), since it controls the relative strength of the positive and negative contributions to  $F_\varepsilon$ . The proofs of Theorems 1.2 and 1.3 rest on new bounds for  $f_\varepsilon$  in  $\mathbf{R}^2 \setminus \mathcal{D}$  and near  $\partial\mathcal{D}$ , which in turn rely on decay estimates for  $\eta_\varepsilon$ . In particular, we show in Lemma 2.4 that  $f_\varepsilon \leq C\varepsilon^{2/3}$  in  $\mathbf{R}^2 \setminus \mathcal{D}$ .

The other part of the proof consists essentially of bounds of  $2\Omega \int \eta_\varepsilon^2 f_\varepsilon Jv$  by the positive terms in  $F_\varepsilon$ . Away from the bulk, we use our estimates of  $f_\varepsilon$  to find that  $2\Omega f_\varepsilon Jv$  is bounded pointwise by  $\frac{1}{2}|\nabla v|^2$ . In the bulk, where  $\eta_\varepsilon^2$  is not too small, we have

$$\frac{1}{2}\eta_\varepsilon^2|\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2 - 1)^2 \geq \eta_\varepsilon^2\left[\frac{1}{2}|\nabla v|^2 + \frac{1}{4\tilde{\varepsilon}^2}(|v|^2 - 1)^2\right]$$

for some  $\tilde{\varepsilon}$  such that  $|\log \tilde{\varepsilon}| = |\log \varepsilon|(1 + o(1))$ . We obtain the desired bounds by combining this with a weighted Jacobian estimate mentioned above, Lemma 4.1, which *directly* implies that

$$2\Omega \int \chi \eta_\varepsilon^2 f_\varepsilon Jv \leq \Omega \left( \frac{2\|f_\varepsilon\|_\infty}{\log \tilde{\varepsilon}} \right) \int \chi \eta_\varepsilon^2 \left[ \frac{1}{2}|\nabla v|^2 + \frac{1}{4\tilde{\varepsilon}^2}(|v|^2 - 1)^2 \right] + \text{small error terms}$$

where  $\chi$  is a cutoff function supported in the bulk. Note that the leading-order critical rotation  $\omega_0$  is such that  $\Omega \left( \frac{2\|f_\varepsilon\|_\infty}{\log \tilde{\varepsilon}} \right) \approx \Omega/\omega_0 |\log \varepsilon|$ . The proof of Theorem 1.3 is completed by assembling these ingredients and controlling error terms. The proof of Theorem 1.2 relies on an additional ingredient, which is that if  $|v| \geq \frac{1}{2}$  in an open set  $U$ , then  $Jv$  is extremely close in  $U$  to  $J(\frac{v}{|v|}) = 0$ . Theorem 1.1 follows immediately from combining Theorems 1.2 and 1.3.

An interesting open problem is to see to what extent this analysis continues to hold if the assumption of radial symmetry is dropped. In our arguments, this symmetry is used heavily in our analysis of the behavior of  $f_\varepsilon$  away from the bulk, and near the boundary of the bulk.

We briefly remark on the assumption (1.7) of quadratic growth. Our proofs show that the absence of vortices in the low density region is a consequence of the fact that the auxiliary function  $f_\varepsilon = \xi_\varepsilon/\eta_\varepsilon^2$  is very small in  $\mathbf{R}^2 \setminus \mathcal{D}$ . The proof of this fact (see Lemma 2.4) can be modified to show that if for example (1.7) holds with  $p < 2$ , then  $f_\varepsilon(r) \geq C\varepsilon r^{1-p/2} \rightarrow \infty$  as  $r \rightarrow \infty$ . However, in this situation  $E_\varepsilon$  is unbounded below for any  $\Omega \neq 0$ . This reflects the fact that a subquadratic trapping potential is not strong enough to contain a rotated condensate.

## 2 Properties of auxiliary functions

In this section we study the real-valued minimizer  $\eta_\varepsilon$  and the auxiliary functions  $\xi_\varepsilon$  and  $f_\varepsilon = \xi_\varepsilon/\eta_\varepsilon^2$  defined as

$$\xi_\varepsilon(r) = \int_r^\infty s \eta_\varepsilon^2(s) ds, \quad f_\varepsilon(r) = \xi_\varepsilon(r)/\eta_\varepsilon^2(r). \quad (2.19)$$

**Theorem 2.1.** *Assume that  $V$  satisfies (1.6), (1.9). Then for every  $\varepsilon > 0$ , there exists a unique positive minimizer  $\eta_\varepsilon$  of  $G_\varepsilon$  in*

$$\mathcal{H} := \{u \in H^1(\mathbf{R}^2) : \int_{\mathbf{R}^2} |u|^2 V(x) < \infty, \int_{\mathbf{R}^2} |u|^2 = 1\}.$$

*Every minimizer of  $G_\varepsilon$  in  $\mathcal{H}$  has the form  $e^{i\alpha}\eta_\varepsilon$ , for  $\alpha$  constant. Moreover,  $\eta_\varepsilon$  is a radial smooth positive function and satisfies (1.14) with*

$$|\lambda_\varepsilon - \lambda_0| \leq C\varepsilon |\log \varepsilon|^{1/2} \quad (2.20)$$

*where  $\lambda_0$  is defined by (1.3). Finally, recall the notations  $R_\delta$  from (1.9) and  $a = \lambda_0 - V$ , the following estimates are satisfied:*

$$\eta_\varepsilon(r) \leq C \varepsilon^{1/6} e^{c\varepsilon^{-1/3}(\sqrt{R}-\sqrt{r})} \quad \text{in } \mathbf{R}^2 \setminus \mathcal{D} \quad (2.21)$$

$$|\eta_\varepsilon - \sqrt{a^+}| \leq C\varepsilon^{1/3} \sqrt{a^+} \quad \text{in } B_{R-\varepsilon^{1/3}} \quad (2.22)$$

$$\|\nabla \eta_\varepsilon\|_{L^\infty(\mathbf{R}^2)} \leq C\varepsilon^{-1}. \quad (2.23)$$

$$\eta'_\varepsilon(r) \leq 0 \quad \text{for all } r \in (R_{-\delta_0}, R_{\delta_0}) \quad (2.24)$$

$$|\eta'_\varepsilon(r)| \leq \frac{C}{\varepsilon} \eta_\varepsilon(r) \sqrt{V(r)} \quad \text{for all sufficiently large } r \quad (2.25)$$

*if  $\varepsilon < \varepsilon_0$ .*

Certain parts of the proof follow quite closely arguments given in [3] and in the pure quadratic case in [11]. Note that some arguments in [11] rely strongly on the special shape of the potential and cannot be generalized to other functions. Since  $V$  is not necessarily increasing, we have property (2.24) only in the neighborhood of  $\partial\mathcal{D}$ .

*Proof. Step 1. Existence of minimizers:* This follows from standard arguments once we notice that  $\int_{\mathbf{R}^2} |u_n|^2 V dx \leq C$  is uniformly bounded for any sequence  $(u_n)$  minimizing  $G_\varepsilon$ , and the set of functions in  $\mathcal{H}$  satisfying such a uniform bound is precompact with respect to weak convergence in  $H^1(\mathbf{R}^2)$ . This last fact is proved by straightforward and well-known arguments, such as are explained in the proof in [11], Lemma 2.1, for  $V$  quadratic, the point being that the bound on  $\int |u|^2 V$  prevents mass escaping to  $\infty$ . Standard theory then implies that any minimizer is smooth. If  $\eta$  is any minimizer, then  $|\eta|$  is as well, since  $G(|\zeta|) \leq G(\zeta)$  for all  $\zeta$ . The strong maximum principle then implies that  $|\eta|$  (and hence  $\eta$ ) never vanishes, and since  $G(\eta) \leq G(|\eta|)$ , it is easy to see that  $\eta/|\eta| = e^{i\alpha}$  for some constant  $\alpha$ . We henceforth let  $\eta_\varepsilon$  denote a fixed positive minimizer.

*Step 2: uniqueness of  $\eta_\varepsilon$ .* Multiplying (1.14) by  $\eta_\varepsilon$  and integrating by parts we find that  $\mu_\varepsilon$  is positive. Suppose that there are two couples  $(\eta_0, \mu_0)$  and  $(\eta_1, \mu_1)$  satisfying (1.14) such that  $\|\eta_0\|_{L^2} = 1 = \|\eta_1\|_{L^2}$  and  $\mu_0 > \mu_1$ , and define  $w = \frac{\eta_1}{\eta_0}$ . This function verify

$$\int_{\mathbf{R}^2} \eta_0^2 (w-1)^2 dx = 2 \int_{\mathbf{R}^2} (\eta_1^2 - \eta_0 \eta_1) dx = 2 \int_{\mathbf{R}^2} \eta_0^2 w(w-1) dx$$

and

$$-\nabla \cdot (\eta_0^2 \nabla w) + \frac{1}{\varepsilon^2} \eta_0^4 w (w^2 - 1) = (\mu_1 - \mu_0) \eta_0^2 w.$$

Multiplying the second equality by  $(w - 1)$ , integrating by parts and then using the first equality we find

$$\int_{\mathbf{R}^2} \left\{ \eta_0^2 |\nabla(w - 1)|^2 + \frac{1}{\varepsilon^2} \eta_0^4 w (w - 1)^2 (w + 1) + \frac{1}{2} (\mu_0 - \mu_1) \eta_0^2 (w - 1)^2 \right\} dx = 0.$$

The integration by parts is justified in view of (2.21), (2.25), which apply to both  $\eta_0$  and  $\eta_1$ , and the proofs of which do not rely on the uniqueness of the minimizer. Hence  $w \equiv 1$  and  $\mu_0 = \mu_1$ .

*Step 3: estimate of  $\lambda_\varepsilon - \lambda_0$ .* We next note, following standard arguments, that  $G_\varepsilon$  can be rewritten

$$G_\varepsilon(\eta) = \int_{\mathbf{R}^2} \left\{ \frac{1}{2} |\nabla \eta|^2 + \frac{1}{4\varepsilon^2} (\eta^2 - a^+)^2 + \frac{1}{2\varepsilon^2} a^- \eta^2 \right\} dx + \frac{1}{2\varepsilon^2} \left( \lambda_0 - \frac{1}{2} \int (a^+)^2 \right)$$

if  $\|\eta\|_2 = 1$ . Let  $G_\varepsilon^1(\eta)$  denote the first integral above. We claim that

$$G_\varepsilon^1(\eta_\varepsilon) \leq C |\log \varepsilon|.$$

Since  $\eta_\varepsilon$  is a minimizer, to prove this it suffices to construct a competitor for which  $G_\varepsilon^1$  is suitably small. To do this, define

$$g_\varepsilon(s) := \begin{cases} \frac{s}{\varepsilon} & \text{if } s \leq \varepsilon^2 \\ \sqrt{s} & \text{if } s \geq \varepsilon^2, \end{cases} \quad \text{and } \tilde{\eta}_\varepsilon := \frac{g_\varepsilon(a^+)}{\|g_\varepsilon(a^+)\|_{L^2}}.$$

Note that

$$1 = \int a^+ \geq \int g_\varepsilon^2(a^+) = \int a^+ - \int_{a^+ \leq \varepsilon^2} a^+ \left\{ 1 - \frac{a^+}{\varepsilon^2} \right\} \geq 1 - C\varepsilon^2.$$

Using this and explicit calculations such as those in [14], Lemma 12, the claim is easily verified. We now multiply (1.14) by  $\eta_\varepsilon$ , integrate by parts and rewrite, recalling the  $L^2$  constraint, to find that

$$\frac{1}{\varepsilon^2} (\lambda_\varepsilon - \lambda_0) = \int |\nabla \eta_\varepsilon|^2 + \frac{1}{\varepsilon^2} (\eta_\varepsilon^2 - (V - \lambda_0)) \eta_\varepsilon^2 dx \quad (2.26)$$

$$\begin{aligned} &= \int |\nabla \eta_\varepsilon|^2 + \frac{1}{\varepsilon^2} (\eta_\varepsilon^2 - a^+ + a^-) \eta_\varepsilon^2 dx \\ &= \int |\nabla \eta_\varepsilon|^2 + \frac{1}{\varepsilon^2} [a^- \eta_\varepsilon^2 + (\eta_\varepsilon^2 - a^+)^2 + (\eta_\varepsilon^2 - a^+) a^+] dx \\ &\leq 4G_\varepsilon^1(\eta_\varepsilon) + \frac{1}{\varepsilon^2} \|\eta_\varepsilon^2 - a^+\|_{L^2} \|a^+\|_{L^2} \leq C[G_\varepsilon^1(\eta_\varepsilon) + \frac{1}{\varepsilon} \sqrt{G_\varepsilon^1(\eta_\varepsilon)}]. \end{aligned} \quad (2.27)$$

Thus we have proved (2.20).



*Step 4 : estimates of  $\eta_\varepsilon$ .*

We claim that

$$\eta_\varepsilon^2 \leq \max_{\mathcal{D}}(\lambda_\varepsilon - V) =: A \quad (2.28)$$

To see this define  $w = \frac{1}{\varepsilon}(\eta_\varepsilon - \sqrt{A})$ . We have that  $\eta_\varepsilon \in L_{loc}^3$ , so after (1.14)  $w, \Delta w \in L_{loc}^1$ . Kato's inequality gives  $\Delta w^+ \geq \text{sgn}^+(w)\Delta w$ . Using (1.14) again we find

$$\Delta w^+ \geq \frac{\text{sgn}^+(w)}{\varepsilon^3} \eta_\varepsilon (\eta_\varepsilon^2 - A) = \frac{\text{sgn}^+(w)}{\varepsilon^3} (\varepsilon w + \sqrt{A}) (\varepsilon^2 w^2 + 2\varepsilon w \sqrt{A}) \geq (w^+)^3 \quad \text{in } \mathcal{D}'$$

Hence we have  $-\Delta w^+ + (w^+)^3 \leq 0$  in  $\mathcal{D}'(\mathbf{R}^2)$  and  $w \in L_{loc}^3$ , so using Lemma 2 in [9],  $w^+ \equiv 0$ .

We remark that the properties of the potential  $V$  at the boundary (1.9) implies that the maximum of  $\lambda_\varepsilon - V$  is attained at an interior point  $x_0$  of  $\mathcal{D}$  such that  $\text{dist}(x_0, \partial\mathcal{D}) > c\delta_0$ .

The minimizer being a solution of (1.14) in  $L^\infty$ , by elliptic regularity we derive that it is a smooth function.

*Proof of (2.21).* We construct a supersolution of (1.14) of the form

$$\bar{\eta}(x) := \begin{cases} \sqrt{\lambda_0 - V(x) + 8\delta} & \text{if } |x| \leq R_{-\delta} \\ \frac{\lambda_0 - \delta - V(x)}{6\sqrt{\delta}} + 3\sqrt{\delta} & \text{if } R_{-\delta} \leq |x| \leq R_\delta \\ \gamma e^{-\frac{\sqrt{|x|}}{\sigma}} & \text{if } R_\delta \leq |x| \end{cases}$$

where  $0 < \delta < \delta_0$  is small parameter that will be determined later and  $\gamma, \sigma$  are chosen such that  $\bar{\eta} \in C^1(\mathbf{R}^2)$ , i.e.,

$$\gamma = \frac{8\sqrt{\delta}}{3} \exp(\sigma^{-1} R_\delta^{1/2}) \quad \text{and} \quad \sigma = 16\delta \frac{R_\delta^{-1/2}}{|\nabla V(R_\delta)|}$$

A straightforward computation shows that for  $\delta = C\varepsilon^{1/3}$ ,  $\bar{\eta}$  is a supersolution of (1.14) and we also have

$$\sigma = \mathcal{O}(\varepsilon^{1/3}) \quad \text{and} \quad \gamma = \mathcal{O}(\varepsilon^{1/6} e^{\varepsilon^{-1/3} \sqrt{R}}).$$

Moreover, with this choice of  $\delta$ ,  $\bar{\eta}^2 > \lambda_\varepsilon - V$  for every  $|x| \leq R_{-\delta}$ , so using (2.28)

$$\eta_\varepsilon^2(x_0) \leq A = \lambda_\varepsilon - V(x_0) < \bar{\eta}(x_0).$$

Because  $\eta_\varepsilon$  and  $\bar{\eta}$  are going to zero at infinity, if the function  $\eta_\varepsilon - \bar{\eta}$  is positive somewhere in  $(r_0, \infty)$ , for  $r_0 := |x_0|$ , then it attains a positive maximum at  $\tilde{r} \in (r_0, \infty)$ , i.e.  $\eta'_\varepsilon(\tilde{r}) = \bar{\eta}'(\tilde{r})$  and  $\eta''_\varepsilon(\tilde{r}) < \bar{\eta}''(\tilde{r})$ . Given the structure of (1.14) and because  $\bar{\eta}$  is a supersolution and  $\eta_\varepsilon$  a solution, if  $V(\tilde{r}) - \lambda_\varepsilon \geq 0$  we would have that  $\eta_\varepsilon(\tilde{r}) \leq \bar{\eta}(\tilde{r})$ . In another hand, if  $V(\tilde{r}) - \lambda_\varepsilon < 0$  then we would have  $\bar{\eta}(\tilde{r}) < \sqrt{\lambda_\varepsilon - V(\tilde{r})}$ , which for  $\varepsilon$  small enough, contradicts the definition of  $\bar{\eta}$ . Hence

$$\eta_\varepsilon(r) \leq \bar{\eta}(r) \quad \text{in} \quad (r_0, \infty).$$

*Proof of (2.22).* Using assumption (1.9), by exactly following [3], one finds that  $|\eta_\varepsilon - \sqrt{a_\varepsilon^+}| \leq C\varepsilon^{1/3}\sqrt{a_\varepsilon^+}$ , for  $a_\varepsilon := \lambda_\varepsilon - V = a + \lambda_\varepsilon - \lambda_0$ . In view of (2.20), this implies (2.22).

*Proof of (2.23).* For  $x \in \mathbf{R}^2$  define  $\tilde{\eta}(y) = \eta_\varepsilon(\varepsilon(y - x))$  in  $B_{2L}(x)$ . This function satisfies

$$\Delta \tilde{\eta} = \tilde{\eta}(V(\varepsilon(y - x)) + \tilde{\eta}^2 - \lambda_\varepsilon) =: h_\varepsilon$$

After estimates (2.21) and (2.22)  $|h_\varepsilon| \leq C$ , so using a Hölder estimate for the first derivative of  $\tilde{\eta}$  (see Theorem 8.32 in [10]) we have that  $\|\nabla \tilde{\eta}\|_{L^\infty(B_L(x))} \leq C$  for a constant  $C$  independent of  $x$  and hence the result.

*Step 4 : Proof of (2.24).*

We denote  $L$  the elliptic operator obtained by linearizing equation (1.14)

$$L := -\Delta + \frac{1}{\varepsilon^2}(V(x) + 3\eta_\varepsilon^2 - \lambda_\varepsilon),$$

and  $\lambda_j$ ,  $j = 1, 2, \dots$ , its eigenvalues in  $\mathbf{R}^2$ .

Let  $\mu$  be the first Dirichlet eigenvalue of  $L$  in the half space  $\Omega = \{x_1 > 0\}$  and  $\psi$  the corresponding eigenfunction (which exists because of the compact embedding of  $\mathcal{H}$  in  $L^2$ ). Since  $V$  and  $\eta_\varepsilon$  are radial, is clear that the odd extension of  $\psi$  to  $\mathbf{R}^2$  is a eigenfunction for  $L$  in  $\mathbf{R}^2$  with corresponding eigenvalue  $\mu = \lambda_j$ . Note that  $j \geq 2$  because the odd extension change sign in  $\mathbf{R}^2$ .

We have that  $L\eta_\varepsilon = 2\eta_\varepsilon^4 > 0$  and  $\eta_\varepsilon > 0$ . Using the maximun principle due to Berestycki, Nirenberg and Varadhan [6], this implies that the first eigenvalue of  $L$  is positive. We will prove that if (2.24) does not hold, then  $\mu < 0$ , which contradicts the fact that  $\lambda_1 > 0$ .

Assume that  $\eta'_\varepsilon(r) > 0$  at some  $r \in (R_{-\delta_0}, R_{\delta_0})$ . Then there exists  $\alpha < r < \beta$  such that  $\eta'_\varepsilon(\alpha) = \eta'_\varepsilon(\beta) = 0$  and  $\eta'_\varepsilon > 0$  in  $(\alpha, \beta)$ . If  $\alpha \leq R_{-2\delta_0}$ , then  $\eta_\varepsilon$  is increasing on  $(R_{-2\delta_0}, R_{-\delta_0})$ , so that  $\eta_\varepsilon(R_{-2\delta_0}) \leq \eta_\varepsilon(R_{-\delta_0})$ . This is impossible for all sufficiently small  $\varepsilon$ , since  $\eta_\varepsilon \rightarrow \sqrt{a^+}$  uniformly for  $r < R_{-\varepsilon^{1/3}}$ , by (2.22), and  $a^+(R_{-2\delta_0}) > a^+(R_{-\delta_0})$ . Thus  $\alpha \geq R_{-2\delta_0}$ . The same argument, but using (2.21) instead of (2.22), shows that  $\beta \leq R_{2\delta_0}$ .

Now let  $D := \{x \in \mathbf{R}^2 : x_1 > 0, \alpha < |x| < \beta\}$ . Then

$$\frac{\partial \eta_\varepsilon}{\partial x_1} > 0 \text{ in } D, \quad \frac{\partial \eta_\varepsilon}{\partial x_1} = 0 \text{ in } \partial D \quad \text{and} \quad L\left(\frac{\partial \eta_\varepsilon}{\partial x_1}\right) = -\frac{\partial V}{\partial x_1}\eta_\varepsilon \leq 0 \text{ in } D.$$

The last inequality come from the differentiation of (1.14) and hypothesis (1.9), which implies that  $\partial V/\partial R > 0$  for  $r \in (R_{-2\delta_0}, R_{2\delta_0})$ . Using the monotonicity of Dirichlet eigenvalues with respect to the domain, this implies that  $\mu < 0$ .

*Step 5 : Proof of (2.25).* For any  $r \geq R$ , define a function  $\tilde{\eta} : (r, \infty) \rightarrow \mathbf{R}$  by

$$\tilde{\eta}(s) := \eta_\varepsilon(r) \exp \left( \alpha \frac{2}{p+2} (s^{\frac{p+2}{2}} - r^{\frac{p+2}{2}}) \right)$$

where  $c_0$  and  $p$  are the constants in (1.7). It follows from (2.20) and (1.7) that if  $s \geq r$  and  $r$  is sufficiently large, then  $V(s) - \lambda_\varepsilon + \tilde{\eta}^2(s) \leq V(s) \leq c_0 s^p$ , so that if  $r$  is sufficiently large, then

$$-\Delta \tilde{\eta} + \frac{1}{\varepsilon^2} (V - \lambda_\varepsilon + \tilde{\eta}^2) \tilde{\eta} \leq -\Delta \tilde{\eta}(s) + \frac{c_0}{\varepsilon^2} s^p \tilde{\eta} = \left( (-\alpha^2 + \frac{c_0}{\varepsilon^2}) s^p + \alpha (\frac{p}{2} + 1) s^{\frac{p}{2}-1} \right) \tilde{\eta}.$$

Choosing  $\alpha = \frac{(2c_0)^{1/2}}{\varepsilon}$ , it follows that  $\tilde{\eta}$  is a subsolution of (1.14) in  $(r, \infty)$  if  $r$  is sufficiently large. For such  $r$ , noting that  $\tilde{\eta}(r) = \eta_\varepsilon(r)$ , we can argue as in the proof of (2.21) to deduce that  $\eta_\varepsilon - \tilde{\eta}$  is nonnegative in  $(r, \infty)$ .

Then since  $\tilde{\eta}(r) = \eta_\varepsilon(r)$  and  $\tilde{\eta}(s) \leq \eta_\varepsilon(s)$  for  $s \geq r$ , we again use (1.7) to conclude that

$$\eta'_\varepsilon(r) \geq \tilde{\eta}'(r) = -\frac{(2c_0)^{1/2}}{\varepsilon} r^{\frac{p}{2}} \eta_\varepsilon(r) \geq -\sqrt{2} \frac{c_0}{\varepsilon} \sqrt{V(r)} \eta_\varepsilon(r)$$

for sufficiently large  $r$ . On the other hand, by choosing  $\alpha = \frac{\sqrt{c_0}}{2\varepsilon}$  in the definition of  $\tilde{\eta}$ , we obtain a decreasing supersolution (still denoted  $\tilde{\eta}$ ) such that  $\tilde{\eta}(r) = \eta_\varepsilon(r)$ . A similar application of the maximum principle shows that  $\eta_\varepsilon$  is bounded above by (the new)  $\tilde{\eta}$  on  $(r, \infty)$ , and in particular this implies that  $\eta'_\varepsilon(r) \leq 0$ . These facts combine to establish (2.25).  $\square$

We next prove

**Lemma 2.2.** *Assume that  $V$  satisfies (1.6) and (1.9) and the quadratic growth condition (1.10). Let  $\eta_\varepsilon$  be the positive minimizer found in Theorem 2.1. Let  $f_\varepsilon(x) := \xi_\varepsilon(x)/\eta_\varepsilon^2(x)$ , where  $\xi_\varepsilon$  was defined in (1.17). Then there exists a constant  $C$  independent of  $\varepsilon \in (0, \varepsilon_1]$  such that*

$$f_\varepsilon(|x|) \leq \begin{cases} C \text{dist}(x, \partial \mathcal{D}) + C\varepsilon^{2/3} & \text{if } x \in \mathcal{D} \\ C\varepsilon^{2/3} & \text{if not.} \end{cases} \quad (2.29)$$

In addition, for all sufficiently small  $\varepsilon$ ,

$$\|\nabla \xi_\varepsilon\|_\infty \leq C \quad (2.30)$$

and

$$\|f_\varepsilon - f_0\|_\infty \leq C\varepsilon^{1/3}. \quad (2.31)$$

*Proof.* For every  $s \geq r \geq R_\delta$  (where  $0 < \delta \leq \delta_0$  will be chosen later), we define

$$\tilde{\eta}(s) = \eta_\varepsilon(r) e^{-\mu_\delta(s^2-r^2)/2} \quad \text{and} \quad \mu_\delta^2 = \frac{c_1(R_\delta^2 - R^2) + (\lambda_\varepsilon - \lambda_0)}{R_\delta^2 \varepsilon^2}. \quad (2.32)$$

Using (1.10), where the constant  $c_1$  is defined, and arguing as in the proof of (2.25), we find that  $\tilde{\eta} - \eta_\varepsilon$  is nonnegative in  $(r, \infty)$ .

We use the previous estimate and the definition of  $\xi_\varepsilon$  to compute

$$f_\varepsilon(r) = \frac{1}{\eta_\varepsilon^2(r)} \int_r^\infty s \eta_\varepsilon^2(s) ds \leq \int_r^\infty e^{-\mu_\delta(s^2-r^2)} s ds = \frac{1}{2\mu_\delta} \quad \text{for } r \geq R_\delta.$$

The definition of  $f_\varepsilon$  implies that  $f'_\varepsilon(r) = -r - 2f_\varepsilon(r) \frac{\eta'_\varepsilon(r)}{\eta_\varepsilon(r)}$ , and from the monotonicity (2.24) of  $\eta_\varepsilon$ , we infer that  $f'_\varepsilon(r) \geq -r$  in  $(R_{-\delta_0}, R_{\delta_0})$ . Thus for any  $R_{-\delta_0} \leq r \leq R_\delta$ ,

$$f_\varepsilon(r) \leq \frac{R_\delta^2 - r^2}{2} + \frac{1}{2\mu_\delta}.$$

We now fix  $\delta = \varepsilon^{2/3}$ , and we conclude from (1.9) and (2.20) that (2.29) holds as long as  $r \geq R_{-\delta_0}$ .

For  $0 \leq r \leq R_{-\delta_0}$ , we write

$$f_\varepsilon(r) = \frac{1}{\eta_\varepsilon^2(r)} \int_r^{R_{-\delta_0}} s \frac{\eta_\varepsilon^2(s)}{\eta_\varepsilon^2(r)} ds + \frac{\eta_\varepsilon^2(R_{-\delta_0})}{\eta_\varepsilon^2(r)} f(R_{-\delta_0})$$

From (2.22) and (1.9), we see that if  $0 \leq r \leq s \leq R_{-\varepsilon^{1/3}}$ , then

$$\frac{\eta_\varepsilon^2(s)}{\eta_\varepsilon^2(r)} \leq \frac{(1 + C\varepsilon^{1/3})^2 a^+(s)}{(1 - C\varepsilon^{1/3})^2 a^+(r)} \leq C \quad \text{for sufficiently small } \varepsilon, \quad (2.33)$$

and by using the and the fact that  $f_\varepsilon(R_{-\delta_0}) \leq C\varepsilon^{2/3} + C\delta_0$ , one easily deduces that (2.29) holds for  $r \in [0, R_{-\delta_0})$ .

Next, the definition of  $\xi_\varepsilon$  implies that  $|\nabla \xi_\varepsilon(x)| = |x| \eta_\varepsilon^2(x)$ , so that (2.30) follows from (2.28) and (2.21).

For  $r \geq R_{-\varepsilon^{1/3}}$ , we see from (2.29) that  $|f_\varepsilon(r) - f_0(r)| \leq C\varepsilon^{1/3} + |f_0(r)|$ . This is trivially bounded by  $C\varepsilon^{1/3}$  if  $r \geq R$ . If  $R \leq r \leq R_{-\varepsilon^{1/3}}$  then (1.9) implies that  $c(R-r) \leq a(r) \leq C(R-r)$ , and thus

$$|f_0(r)| = f_0(r) \leq \frac{C}{r-R} \int_r^R s(R-s) ds \leq C(R-r) \leq C\varepsilon^{1/3}.$$

For  $0 \leq r \leq R_{-\varepsilon^{1/3}}$  we write

$$\begin{aligned} f_\varepsilon(r) - f_0(r) &= \left\{ \frac{1}{\eta_\varepsilon^2(r)} \int_r^{R_{-\varepsilon^{1/3}}} s \eta_\varepsilon^2(s) ds - \frac{1}{a(r)} \int_r^{R_{-\varepsilon^{1/3}}} s a(s) ds \right\} \\ &\quad + \frac{\eta_\varepsilon^2(R_{-\varepsilon^{1/3}})}{\eta_\varepsilon^2(r)} f_\varepsilon(R_{-\varepsilon^{1/3}}) - \frac{a(R_{-\varepsilon^{1/3}})}{a(r)} f_0(R_{-\varepsilon^{1/3}}) \\ &= I + II - III \end{aligned}$$

Using (2.33) and our earlier estimates of  $f_\varepsilon, f_0$  for  $r \geq R_{-\varepsilon^{1/3}}$ , we see that

$$|II| \leq C f_\varepsilon(R_{-\varepsilon^{1/3}}) \leq C \varepsilon^{1/3} \quad \text{and} \quad |III| \leq C f_0(R_{-\varepsilon^{1/3}}) \leq C \varepsilon^{1/3}.$$

We further decompose the remaining term as

$$I = \left( \frac{1}{\eta_\varepsilon^2(r)} - \frac{1}{a(r)} \right) \int_r^{R_{-\varepsilon^{1/3}}} s \eta_\varepsilon^2(s) ds + \frac{1}{a(r)} \int_r^{R_{-\varepsilon^{1/3}}} s (\eta_\varepsilon^2(s) - a(s)) ds.$$

Using (2.22), it follows that

$$|I| \leq C \varepsilon^{1/3} \int_r^{R_{-\varepsilon^{1/3}}} s \frac{\eta_\varepsilon^2(s)}{\eta_\varepsilon^2(r)} ds + C \varepsilon^{1/3} \int_r^{R_{-\varepsilon^{1/3}}} s \frac{a(s)}{a(r)} ds.$$

Due to (2.24),  $\frac{\eta_\varepsilon^2(s)}{\eta_\varepsilon^2(r)} \leq 1$  if  $R_{-\delta_0} \leq r \leq s \leq R_{-\varepsilon^{1/3}}$ . And if  $0 \leq r \leq R_{-\delta_0}$  then  $\eta_\varepsilon^2(r) \geq C^{-1}$  and so  $\frac{\eta_\varepsilon^2(s)}{\eta_\varepsilon^2(r)} \leq C$ . Thus the first integral is bounded by  $C \varepsilon^{1/3}$ . The second integral is similarly estimated, using (1.9) in place of (2.24).  $\square$

**Remark 2.3.** *In the case of a potential  $V$  for which (1.8) fails, so that for example  $\mathcal{D}$  has the form  $B_R \setminus B_{R'}$ , one expects that instead of being small,  $f_\varepsilon$  is large, namely,  $f_\varepsilon \geq c e^{c/\varepsilon}$  in the interior of  $B_{R'}$ . This is related to the formation at very low rotations of a giant vortex in the interior of  $B_{R'}$ . The arguments used to prove Lemma 2.4 show in this situation that if  $V$  grows quadratically in the complement of  $B_R$ , as in (1.10), then  $f_\varepsilon$  is very small in  $\mathbf{R}^2 \setminus B_R$ . This suggests that at low rotations there should be no vortices in  $\mathbf{R}^2 \setminus B_R$ , but this cannot be deduced from the arguments we use to prove Theorems 1.2 and 1.3.*

The last lemma in this section examines the case when  $V$  has subquadratic growth and  $f_\varepsilon$  is also large so that in principle vortices could exist in the low density region.

**Lemma 2.4.** *Assume that  $V$  satisfies (1.6), (1.9) and*

$$\text{there exists } c_2 > 0 \text{ and } p < 2 \text{ such that } V(r) \leq c_2(r^p + 1) \text{ for all } r \geq R. \quad (2.34)$$

*Then  $f_\varepsilon(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$ .*

Note that with these assumptions on  $V$ , there is a sequence of functions  $\zeta_\alpha$  in  $\mathcal{H}$  such that  $\inf_\alpha G_\varepsilon(\zeta_\alpha) = -\infty$ . Physically this happens because the centrifugal force due to rotation is bigger than the subquadratic trapping potential. This indicates that, although one can prove that in this situation,  $f_\varepsilon \rightarrow \infty$  as  $r \rightarrow \infty$  (compare Lemma 2.4), this is not expected to give any information about the physical behaviour of condensates.

*Proof.* Let  $q > 2$ . For every  $r \geq \max\{1, R\}$ , we claim that

$$\eta_\varepsilon(s) \geq \eta_\varepsilon(r) e^{-\nu_{\varepsilon,r}(s^q - r^q)/q} \quad (2.35)$$

for all  $s \geq r$ . Where  $\nu_{\varepsilon,r}$  is the positive root of the polynomial  $\nu^2 - \frac{q}{r^q} \nu - \frac{c}{\varepsilon^2 r^{2q-2-p}}$ , which for  $\varepsilon$  small satisfy

$$\nu_{\varepsilon,r} < C \varepsilon^{-1} r^{-\beta}$$

with  $\beta = q - 1 - p/2$ . Indeed, the right hand side of (2.35) is a subsolution in  $(r, \infty)$  of (1.14) while  $\eta_\varepsilon$  is a solution. Both functions are going to zero at infinity and they are equal at  $s = r$ , so the result comes arguing as in the proof of (2.21).

We use the previous estimates and the definition of  $\xi_\varepsilon$  to compute

$$f_\varepsilon(r) = \frac{\xi_\varepsilon(r)}{\eta_\varepsilon^2(r)} = \frac{1}{\eta_\varepsilon^2(r)} \int_r^\infty s \eta_\varepsilon^2(s) ds \geq \int_r^\infty e^{-\nu_r(s^q - r^q)} s ds \geq \frac{r^{2-q}}{\nu_r} > C \varepsilon r^{1-p/2}.$$

and hence the result.  $\square$

### 3 Splitting the energy

In this section we recall the proofs of (1.15) and (1.18).

For  $U \subset \mathbf{R}^2$ , we will write  $E_\varepsilon(w; U)$  etc to denote the integrals over  $U$  of the energy density appearing in the definition of  $E_\varepsilon(u) = E_\varepsilon(u; \mathbf{R}^2)$ , and similarly  $G_\varepsilon(\cdot, U), F_\varepsilon(\cdot, U)$ .

Note that  $v = u/\eta_\varepsilon$  is well defined since  $\eta_\varepsilon > 0$ . Since  $\eta_\varepsilon$  satisfies (1.14), we multiply it by  $\eta_\varepsilon(1 - |v|^2)$  and integrate over a ball  $B_r$  to find that

$$\int_{B_r} (|v|^2 - 1) \left( -\frac{1}{4} \Delta \eta_\varepsilon^2 + \frac{1}{2\varepsilon^2} \eta_\varepsilon^2 (V(x) + \eta_\varepsilon^2) + \frac{1}{2} |\nabla \eta_\varepsilon|^2 \right) = \frac{\lambda_\varepsilon}{\varepsilon^2} \int_{B_r} (|u|^2 - \eta_\varepsilon^2).$$

Note that the Lagrange multiplier term tends to 0 as  $r \rightarrow \infty$ , since both the  $L^2$  norms of  $u$  and  $\eta_\varepsilon$  are 1. Moreover,

$$\begin{aligned} E_\varepsilon(v\eta_\varepsilon; B_r) &= J_\varepsilon(\eta_\varepsilon; B_r) + F_\varepsilon(v; B_r) + \int_{B_r} \frac{1}{2} |\nabla \eta_\varepsilon|^2 (|v|^2 - 1) + \frac{1}{2} \eta_\varepsilon \nabla \eta_\varepsilon \cdot \nabla |v|^2 \\ &\quad - \frac{1}{4\varepsilon^2} \eta_\varepsilon^4 (1 - |v|^2)^2 + \frac{1}{4\varepsilon^2} \eta_\varepsilon^4 |v|^4 + \frac{1}{2\varepsilon^2} V(x) \eta_\varepsilon^2 |v|^2 - \frac{1}{4\varepsilon^2} \eta_\varepsilon^4 - \frac{1}{2\varepsilon^2} V(x) \eta_\varepsilon^2. \end{aligned}$$

We integrate by parts to obtain

$$\int_{B_r} \frac{1}{2} \eta_\varepsilon \nabla \eta_\varepsilon \cdot \nabla |v|^2 = - \int_{B_r} \frac{1}{4} |v|^2 \Delta \eta_\varepsilon^2 + \int_{\partial B_r} \frac{1}{2} |v|^2 \eta_\varepsilon \nu \cdot \nabla \eta$$

We use (2.25) to estimate

$$\left| \int_{\partial B_r} \frac{1}{2} |v|^2 \eta_\varepsilon \nu \cdot \nabla \eta \right| \leq \frac{C}{\varepsilon} \int_{\partial B_r} \frac{1}{2} \eta^2 |v|^2 \sqrt{V} = \frac{C}{\varepsilon} V(r)^{-1/2} \int_{\partial B_r} V |u|^2.$$

Since  $\int_{\mathbf{R}^2} V |u|^2 < \infty$ , we can easily find a sequence  $r_k \rightarrow \infty$  such that the above integral tends to 0. Combining the above and letting  $r_k \rightarrow \infty$  along this sequence, we obtain (1.15).

The only property of  $V$  that the above argument used (implicitly) was (1.7), which will be used in the proof of (2.25).

The integration by parts that leads to (1.18) is justified in a similar fashion. One must estimate boundary terms of the form  $\int_{\partial B_r} \xi \nu \cdot (iv, \nabla v)$ . To do this we note that

$$\xi \nu \cdot (iv, \nabla v) = f_\varepsilon(r) \eta_\varepsilon^2 (iv, \nabla v) = f_\varepsilon(r) (iu, \nabla u) \leq \|f_\varepsilon\|_\infty (|u|^2 + |\nabla u|^2).$$

We prove in (2.29) that  $f_\varepsilon$  is bounded as long as  $V$  satisfies (1.10) (in fact we show that  $f_\varepsilon \leq C\varepsilon^{2/3}$  for large  $r$ ) and since  $u \in H^1(\mathbf{R}^2)$ , we can again find a sequence  $r_k \rightarrow \infty$  such that the boundary terms vanish.

Note also that the fact that  $f_\varepsilon \in L^\infty$ , or equivalently that  $|\xi_\varepsilon| \leq C\eta_\varepsilon^2$ , implies that the term  $\xi_\varepsilon Jv$  appearing in (1.18) is integrable on  $\mathbf{R}^2$  for  $v = u/\eta_\varepsilon$ , whenever  $u$  has finite energy.

## 4 Proofs of Theorems 1.2 and 1.3

In this section we use the estimates we have already established to complete the proofs of our main theorems.

*Proof of Theorem 1.2.* We assume that  $u_\varepsilon$  minimizes  $E_\varepsilon$  and that  $\Omega \leq C|\log \varepsilon|$  is such that (1.11) holds.

Let  $\chi$  be a smooth function such that  $\chi \equiv 1$  in  $\{x \in \mathcal{D} : \text{dist}(x, \partial\mathcal{D}) \geq 2|\log \varepsilon|^{-3/2}\}$ , and with support in  $\mathcal{D}_1$ . We also assume that  $\|\nabla \chi\|_\infty \leq 2|\log \varepsilon|^{3/2}$ .

Let  $v = u_\varepsilon/\eta_\varepsilon$ , so that  $E_\varepsilon(u) = G_\varepsilon(\eta_\varepsilon) + F_\varepsilon(v) = E_\varepsilon(\eta_\varepsilon) + F_\varepsilon(v)$ . Thus  $F_\varepsilon(v) \leq 0$ . We write

$$F_\varepsilon(v) = A_1 - A_2 + B$$

where

$$A_1 = \int_{\mathbf{R}^2} \chi \left[ \frac{\eta_\varepsilon^2}{2} |\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 \right] dx, \quad A_2 = 2\Omega \int_{\mathbf{R}^2} \chi \xi_\varepsilon Jv dx$$

and

$$B = \int_{\mathbf{R}^2} (1 - \chi) \left[ \frac{\eta_\varepsilon^2}{2} (|\nabla v|^2 - 4\Omega f_\varepsilon Jv) + \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 \right] dx,$$

It follows directly from our estimates on  $f_\varepsilon$  that  $0 < f_\varepsilon \leq C(\varepsilon^{2/3} + |\log \varepsilon|^{-3/2})$  in the support of  $1 - \chi$ , for small enough  $\varepsilon$ . Since  $\Omega \leq C|\log \varepsilon|$ , it follows that  $\Omega f_\varepsilon \leq \frac{1}{4}$  for all sufficiently small  $\varepsilon$  and (recalling that  $|Jv| \leq \frac{1}{2}|\nabla v|^2$ ) we deduce that

$$(|\nabla v|^2 - 4\Omega f_\varepsilon Jv) \geq \frac{1}{2}|\nabla v|^2$$

in the support of  $1 - \chi$ . It follows immediately that

$$B \geq \int_{\mathbf{R}^2} (1 - \chi) \left[ \frac{\eta_\varepsilon^2}{4} |\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 \right] dx \geq 0 \quad (4.36)$$

and hence that  $B = 0$  if and only if  $v$  is a constant of modulus 1 in the support of  $1 - \chi$ .

Since  $F_\varepsilon(v) \leq 0$ , it is clear that  $A_1 + B \leq A_2$ .

Next, define  $\tilde{\varepsilon} = \varepsilon / (\inf_{\mathcal{D}_1} \eta_\varepsilon)$ , so that (in view of (2.22) and the definition of  $\mathcal{D}_1$ )

$$\tilde{\varepsilon} \leq C\varepsilon |\log \varepsilon|^{3/4}, \quad \frac{1}{\tilde{\varepsilon}^2} \leq \frac{\eta_\varepsilon^2}{\varepsilon^2} \text{ in } \mathcal{D}_1.$$

Then (4.36) and (2.22) imply that,

$$\int_{\mathcal{D}_1} \frac{1}{2} |\nabla v|^2 + \frac{1}{4\tilde{\varepsilon}^2} (|v|^2 - 1)^2 \leq (\inf_{\mathcal{D}_1} \eta_\varepsilon)^{-2} (A_1 + 2B) \leq C |\log \varepsilon|^{3/2} A_2. \quad (4.37)$$

To continue, let  $w = \frac{v}{|v|} = w^1 + iw^2$ . From (1.11) we see that  $|v| \geq \frac{1}{2}$  in  $\mathcal{D}_1$ , and hence it is clear that  $w \in H^1(\mathcal{D}_1)$ , and  $|w|^2 \equiv 1$ . It follows that  $Jw = 0$ ; we will recall a standard proof of this fact in a moment. Thus

$$A_2 = 2\Omega \int_{\mathcal{D}_1} \chi_{\xi_\varepsilon} (Jv - Jw) \, dx = 2\Omega \int_{\mathcal{D}_1} \nabla^\perp(\chi_{\xi_\varepsilon}) \cdot [(iv, \nabla v) - (iw, \nabla w)] \, dx.$$

If we write  $v = \rho e^{i\phi}$  in  $\mathcal{D}_1$ , then a calculation shows that

$$(iv, \nabla v) = \rho^2 \nabla \phi, \quad (iw, \nabla w) = \nabla \phi.$$

From the latter fact we see that  $Jw = \frac{1}{2} \nabla \times (iw, \nabla w) = 0$ , as we asserted above. Also, from this and the fact that  $\rho \geq \frac{1}{2}$  in  $\mathcal{D}_1$  we estimate

$$|(iv, \nabla v) - (iw, \nabla w)| = \frac{|\rho^2 - 1|}{\rho} |\rho \nabla \phi| \leq 2 |v|^2 - 1 |\nabla v|.$$

Using (4.37), we deduce that

$$\begin{aligned} A_2 &\leq 2\Omega \|\nabla(\chi_{\xi_\varepsilon})\|_\infty \int_{\mathcal{D}_1} \left( \frac{\tilde{\varepsilon}}{2} |\nabla v|^2 + \frac{1}{2\tilde{\varepsilon}} (|v|^2 - 1)^2 \right) dx \\ &\leq C\Omega \|\nabla(\chi_{\xi_\varepsilon})\|_\infty \varepsilon |\log \varepsilon|^{9/4} A_2. \end{aligned}$$

One checks easily from the definitions and from (2.30) that

$$\|\nabla(\chi_{\xi_\varepsilon})\|_\infty \leq \|\nabla \chi\|_\infty \|\xi_\varepsilon\|_\infty + \|\nabla \xi_\varepsilon\|_\infty \leq C |\log \varepsilon|^{3/2} \quad (4.38)$$

so we conclude that  $A_2 \leq C\varepsilon |\log \varepsilon|^{15/4} A_2 \leq \frac{1}{2} A_2$  for all sufficiently small  $\varepsilon$ . We know from (4.37) that  $A_2 \geq 0$ , and it follows that  $A_2 = 0$ , and hence (again appealing to (4.37)) that  $A_1 = B = 0$ . Thus  $\|\nabla v\|_{L^2} = \|1 - |v|^2\|_{L^2} = 0$ , and so  $v$  is a constant of modulus 1 as required.  $\square$

The proof of Theorem 1.3 will use the following result, which is Lemma 8 in [14].

**Lemma 4.1.** *There exists a universal constant  $C > 0$  such that for any  $\kappa \in (1, 2)$ , open set  $U \subset \mathbf{R}^2$  and  $u \in H^1(U; \mathbf{R}^2)$ , and  $\varepsilon \in ((0, 1)$ ,*

$$\begin{aligned} |\int_U \phi Ju| &\leq \kappa \int |\phi| \frac{e_\varepsilon(u)}{|\log \varepsilon|} \\ &\quad + C\varepsilon^{(\kappa-1)/50} (1 + \|\phi\|_{W^{1,\infty}}) \left( \|\phi\|_\infty + 1 + \int_{\text{supp } \phi} (|\phi| + 1) e_\varepsilon(u) \, dx \right) \end{aligned} \quad (4.39)$$

for all  $\phi \in C_c^{0,1}(U)$ . Here  $e_\varepsilon(u) = \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (|u|^2 - 1)^2$ .



The lemma as stated in [14] does not explicitly specify the exponent  $(\kappa - 1)/50$  appearing on the right-hand side of (4.39). By inspection of the proof, however, one sees that this exponent can be taken to have the form  $\frac{1}{2}\alpha$ , where  $\alpha = (\kappa - 1)/12\kappa$  as in Theorem 2.1 of [15].

*Proof of Theorem 1.3.* We continue to use notation from the proof of Theorem 1.2, such as  $A_1, A_2, B, \tilde{\varepsilon}$ , and so on.

We first invoke the lemma, with  $\tilde{\varepsilon}$  in place of  $\varepsilon$  and  $\chi \xi_\varepsilon$  in place of  $\phi$ , and with  $\kappa > 1$  to be chosen. This yields

$$|A_2| \leq 2\Omega\kappa \int_{\mathbf{R}^2} \chi \xi_\varepsilon \frac{e_{\tilde{\varepsilon}}(v)}{|\log \tilde{\varepsilon}|} dx + \mathcal{E},$$

where  $\mathcal{E}$  denotes the error terms in (4.39). We note that for all sufficiently small  $\varepsilon > 0$ , the error term satisfies the bound  $\mathcal{E} \leq C\varepsilon^\beta(1 + |A_2|)$ , for  $\beta = (\kappa - 1)/100$ , for all sufficiently small  $\varepsilon$ . This is a consequence of (4.37) and the estimates

$$\|\chi \xi_\varepsilon\|_{W^{1,\infty}} \leq C|\log \varepsilon|^{3/2}, \quad \|\chi \xi_\varepsilon\|_{L^\infty} \leq C.$$

These in turn follow from (4.38) together with (2.30).

Now the choice of  $\tilde{\varepsilon}$  implies that  $e_{\tilde{\varepsilon}}(v) \leq \frac{1}{2}|\nabla v|^2 + \frac{\eta_\varepsilon^2}{4\varepsilon^2}(|v|^2 - 1)^2$  in  $\mathcal{D}_1$ , and recalling that  $\xi_\varepsilon = f_\varepsilon \eta_\varepsilon^2$ , we obtain

$$\begin{aligned} (1 - C\varepsilon^\beta)|A_2| &\leq 2\Omega\kappa \frac{\|f_\varepsilon\|_\infty}{|\log \tilde{\varepsilon}|} \int \chi \left( \frac{\eta_\varepsilon^2}{2} |\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 \right) + C\varepsilon^\beta \\ &= 2\Omega\kappa \frac{\|f_\varepsilon\|_\infty}{|\log \tilde{\varepsilon}|} A_1 + C\varepsilon^\beta. \end{aligned}$$

We know from (2.31) that  $\|f_\varepsilon\|_\infty \leq (1 + C\varepsilon^{1/3})\|f_0\|_\infty \leq (1 + C\varepsilon^\beta)\|f_0\|_\infty$ , and from the choice of  $\tilde{\varepsilon}$ , for any  $K > 0$  there exists  $\varepsilon_0 > 0$  such that  $|\log \tilde{\varepsilon}| \geq (|\log \varepsilon| - \log |\log \varepsilon|)(1 + K\varepsilon^\beta)$  if  $0 < \varepsilon < \varepsilon_0$ . Thus

$$|A_2| \leq \Omega \left( \frac{2\|f\|_\infty}{|\log \varepsilon| - \log |\log \varepsilon|} \right) \kappa A_1 + C\varepsilon^\beta$$

for all sufficiently small  $\varepsilon$ . Assume that  $\Omega \leq \frac{1}{2\|f\|_\infty}(|\log \varepsilon| - (c_1 + 1)\log |\log \varepsilon|)$ , for  $c_1$  to be chosen below. Then

$$|A_2| \leq \left( 1 - c_1 \frac{\log |\log \varepsilon|}{|\log \varepsilon| - \log |\log \varepsilon|} \right) \kappa A_1 + C\varepsilon^\beta \leq \left( 1 - c_1 \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right) \kappa A_1 + C\varepsilon^\beta. \quad (4.40)$$

We now take  $\kappa := 1 + c_1 \frac{\log |\log \varepsilon|}{|\log \varepsilon|}$ , so that  $\beta = (\kappa - 1)/100 = \frac{c_1}{100} \frac{\log |\log \varepsilon|}{|\log \varepsilon|}$ . Recalling that  $A_1 + B \leq A_2$  and that  $B \geq 0$ , clearly  $A_1 \leq A_2$ , so we deduce that

$$c_1^2 \left( \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right)^2 A_1 \leq C\varepsilon^\beta = C|\log \varepsilon|^{-c_1/100}.$$

If  $c_1 = 400$  then we conclude that  $A_1 \leq C|\log \varepsilon|^{-2}$ .

Then (4.40) implies that  $A_2 \leq C|\log \varepsilon|^{-2}$ , and it follows that  $B \leq C|\log \varepsilon|^{-2}$ . In view of (4.37), this implies that

$$\int_{\mathcal{D}_1} |\nabla v|^2 + \frac{1}{4\varepsilon^2}(|v|^2 - 1)^2 \leq C|\log \varepsilon|^{-2}. \quad (4.41)$$

The estimate  $\|\nabla v\|_\infty \leq \frac{C}{\varepsilon}$  (see (2.23)) and (4.41) are easily seen to imply that

$$|v| \geq 1 - C|\log \varepsilon|^{-1} \text{ in } \mathcal{D}_1 \quad (4.42)$$

for all sufficiently small  $\varepsilon$ . Thus  $\Omega$  is subcritical for small enough  $\varepsilon$ . □

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